# $e^+ + e^- \rightarrow \pi^+ + \pi^- + \pi^0$ and Overlapping Resonances*

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The process $e^+ + e^- \rightarrow \pi^+ + \pi^- + \pi^0$ is suggested as a useful tool for studying overlapping resonances. A simple model is used to estimate the effects of the two-pion interactions upon the form of the invariant amplitude. Except for a common factor, which depends only upon the total energy, the two-pion effects occur additively in the resulting form in contrast to the multiplicative effects suggested by other authors.

## I. INTRODUCTION

SEVERAL authors[1–7] have recently suggested approximation schemes in which two particles in a three-particle state are treated as a single unstable particle (corresponding to a resonance in the two-particle system), reducing the three-particle problem to the much simpler two-particle case. Practical applications of these schemes are complicated by the plethora of two-particle resonances: It is difficult to choose an interesting three-particle system in which only one pair of particles interacts strongly. For example, in the $\pi\pi N$ system both $\pi N$ combinations as well as the $\pi\pi$ system can resonate. The complications arise largely because an angular momentum decomposition suitable for one pairing is not at all suited to the other two.

This problem of overlapping resonances has recently been considered in a simple model by Peierls and Tarski.[8] Their model gives simple results only near the static limit and, therefore, does not give a great deal of insight into cases of physical interest where the overlapping usually occurs at fairly high energy. The Lee model, in which exact solutions for the $V+\theta \rightarrow N+\theta+\theta$ and $N+\theta+\theta \rightarrow N+\theta+\theta$ amplitudes can be obtained,[9,10] is similarly deficient.

The purpose of this paper is to suggest the process $e^+ + e^- \rightarrow \pi^+ + \pi^- + \pi^0$ as being particularly amenable to a combined experimental and theoretical attack on the three-particle problem, especially with regard to overlapping resonances. Colliding beams of sufficient energy should soon be available, and, providing the cross-section is not too low, detailed experimental data should then be readily obtainable. The analysis of this process is relatively simple since the final state consists of three spinless identical (except for charge) particles which, to lowest order in the electromagnetic interaction, must be in a $J=1$ state.

The cross section depends upon a single highly symmetric function of three independent variables, defined in Sec. II, in contrast to the several invariant functions of five variables required for a complete analysis of most other experimentally observable three-particle processes. In Sec. III we use a simple model to estimate the effects of the $\pi\pi$ interaction, assumed dominated by the presence of the $\rho$ resonance, upon this function. The results are discussed and compared with the work of other authors in Sec. IV.

## II. KINEMATICS

The process $e^+ + e^- \rightarrow \pi^+ + \pi^- + \pi^0$ has been discussed by Cabibbo and Gatto[11] and by Tsai.[12] To lowest order in the electromagnetic interaction, it is completely determined by a single invariant function, $H(s;s_+,s_-,s_0)$, defined by

$$\langle \pi^+\pi^-\pi^0 | j^\mu(0) | 0 \rangle = (8\omega_+\omega_-\omega_0)^{-1/2} \epsilon^{\mu\nu\rho\sigma} p_\nu^{(+)} p_\rho^{(-)} p_\sigma^{(0)} \times H(s;s_+,s_-,s_0), \quad (1)$$

where $j^\mu(x)$ is the electromagnetic current operator,

$$s = (p^{(+)} + p^{(-)} + p^{(0)})^2 \quad (2)$$

and

$$s_+ = (p^{(-)} + p^{(0)})^2 = s + \mu^2 - 2s^{1/2}\omega_+, \quad (3)$$

$\omega_+$ being the energy of the positive pion in the total center-of-mass system, with analogous relations for $s_-$ and $s_0$. The four variables are related by

$$s_+ + s_- + s_0 = s + 3\mu^2, \quad (4)$$

and $H(s;s_+,s_-,s_0)$ must be symmetric under interchange of any two of $s_+$, $s_-$, and $s_0$.

With $H(s;s_+,s_-,s_0)$ defined by (1), the cross section is given by[11,12]

$$\frac{d^3\sigma}{d\cos\theta d\omega_+ d\omega_-} = \frac{\alpha}{(2\pi)^2} \frac{1}{64E^2} \sin^2\theta |\mathbf{p}^{(+)} \times \mathbf{p}^{(-)}|^2 \times |H(s;s_+,s_-,s_0)|^2, \quad (5)$$

where $E = \frac{1}{2}s^{1/2}$ is the center-of-mass energy of the electron, $\theta$ is the angle between the electron beam and

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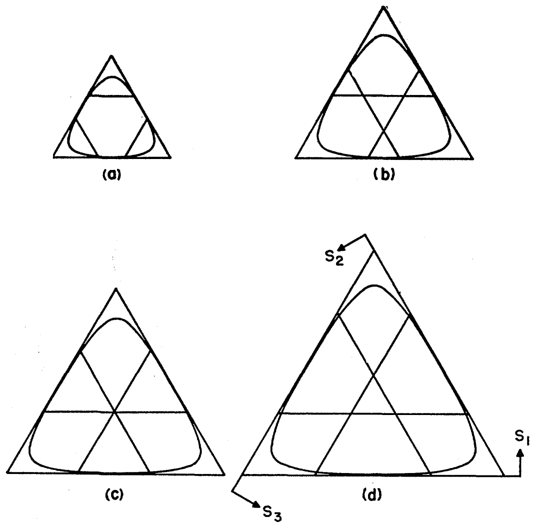


FIG. 1. Phase space for the process $e^+ + e^- \to \pi^+ + \pi^- + \pi^0$, showng the lines $s_i = m_\rho^2 \approx 29\mu^2$. Diagrams (a), (b), (c) and (d) are for $s = 50$, 70, 84, and $100\mu^2$ ($E \approx 495$, 585, 640, and 700 MeV), respectively. The bounding triangles are formed by the lines $s_i = 4\mu^2$ and have altitudes $s - 9\mu^2$.

the normal to the plane of pion production, and $\alpha = e^2/(4\pi)$ is the fine structure constant. Transforming to invariant variables, we find

$$\frac{d^4\sigma}{d\cos\theta ds_+ ds_- ds_0} = \frac{\alpha}{(2\pi)^2(12)^3}\sin^2\theta\left\{\frac{27[s_+s_-s_0 - \mu^2(s-\mu^2)^2]}{s^3}\right\} \times |H(s;s_+,s_-,s_0)|^2\delta(s_+ + s_- + s_0 - s - 3\mu^2), \quad (6)$$

which is a convenient form for comparison with the experimental distribution of events on a Dalitz plot (see Fig. 1). The boundary of phase space is determined by the equation $s_+s_-s_0 = \mu^2(s-\mu^2)^2$.

The "phase space" distribution for the process, obtained by setting $H(s;s_+,s_-,s_0)$ equal to a constant, has contour lines $s_+s_-s_0 = C$, $\mu^2(s-\mu^2)^2 < C < \frac{1}{3}(s+3\mu^2)$. We expect, however, that the distribution will be modified by the existence of the $\rho$ resonance in the two-pion systems; it should be peaked about the lines $s_i = m_\rho^2$ which are shown in Fig. 1. We see that there may be striking effects due to the crossing of the resonance bands. These bands should first appear for $s = (m_\rho + \mu)^2$ ($E \approx 450$ MeV) and will first cross when $s = 2m_\rho^2 + \mu^2$ ($E \approx 540$ MeV), with the triple crossing [Fig. 1(c)] occurring when $s = 3(m_\rho^2 - \mu^2)$ ($E \approx 640$ MeV).

### III. A SIMPLE MODEL FOR $H(s;s_+,s_-,s_0)$

The amplitude $H(s;s_1,s_2,s_3)$ has previously been studied in connection with the isoscalar nucleon structure[4] and the photoproduction of pions from nucleons[13,14]; it is also closely related to the amplitude for $\tau$ decay.[15,16] We shall here follow closely the work of Gourdin and Martin,[13] which we attempt to extend to photon masses above the three-pion threshold. As has been pointed out by Bonnevay[17] and by Barton and Kacser,[18] the arguments which Gourdin and Martin use to justify their approximate integral equation are no longer valid in this region. The calculations below, therefore, must be regarded as applying to a model, similar in many respects to that of Peierls and Tarski,[8] which may retain enough of the essential features to provide a qualitative description of $H(s;s_1,s_2,s_3)$.

We begin by assuming the relation

$$H(s;s_1,s_2,s_3) = \sum_{i=1}^{3}\frac{1}{\pi}\int_{4\mu^2}^{\infty}\frac{\rho(s;x)dx}{x - s_i - i\epsilon}, \quad (7)$$

where

$$\rho(s;x) = e^{-i\delta(x)}\sin\delta(x)f(s;x). \quad (8)$$

Here, $\delta(x)$ is the $T=1$, $J=1$ phase shift for $\pi\pi$ scattering, while

$$f(s;x) = \frac{3}{4}\int_{-1}^{1}\sin^2\theta d\cos\theta \times H[s;x,s_2(x,\cos\theta),s_3(x,\cos\theta)], \quad (9)$$

with

$$\begin{aligned} s_2(x,\cos\theta) &= \tfrac{1}{2}[s + 3\mu^2 - x + 4p(s;x)q(x)\cos\theta], \\ s_3(x,\cos\theta) &= \tfrac{1}{2}[s + 3\mu^2 - x - 4p(s;x)q(x)\cos\theta], \end{aligned} \quad (10)$$

where

$$p^2(s;x) = (4x)^{-1}[s - (x^{1/2}+\mu)^2][s - (x^{1/2}-\mu)^2], \quad (11)$$

and

$$q^2(x) = \tfrac{1}{4}(x - 4\mu^2). \quad (12)$$

Taking the angular projection of the equation for $H$, we obtain a singular integral equation for $f(s;x)$:

$$f(s;x) = \frac{1}{\pi}\int_{4\mu^2}^{\infty}dx'\,\frac{e^{-i\delta(x')}\sin\delta(x')f(s;x')}{x' - x - i\epsilon} + \frac{3}{4}\int_{-1}^{1}\sin^2\theta d\cos\theta\,\frac{1}{\pi}\int_{4\mu^2}^{\infty}dx' e^{-i\delta(x')}\sin\delta(x')f(s;x') \times\left[\frac{1}{x' - s_2(x,\cos\theta) - i\epsilon} + \frac{1}{x' - s_3(x,\cos\theta) - i\epsilon}\right]. \quad (13)$$

If we consider the integrals in (13) as boundary values of analytic functions, then the first integral produces the usual elastic cut running along the real axis from $4\mu^2$ to $\infty$. The function corresponding to the second

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integral, however, has branch points at $-\infty$, 0, and $\frac{1}{2}(s-\mu^2)$ connected by cuts which wind tortuously through the complex plane. For $s>9\mu^2$, these cuts overlap and entangle the elastic cut, making the analysis in terms of the boundary value of a single analytic function quite difficult. We can avoid most of these difficulties and yet retain the feature of overlapping cuts by making an approximation suggested by Gourdin and Martin[13]: We replace $\frac{3}{4}\sin^2\theta$ by $\delta(\cos\theta)$, obtaining the approximate, but much simpler integral equation

$$f(s;x)=\frac{1}{\pi}\int_{4\mu^2}^{\infty}dx' e^{-i\delta(x')}\sin\delta(x')f(s;x')\times\left[\frac{1}{x'-x-i\epsilon}+\frac{2}{x'-\frac{1}{2}(s+3\mu^2-x)-i\epsilon}\right]. \quad (14)$$

This equation is deceptively similar to the exactly soluble equation which occurs in the Lee model but lacks the essential symmetry between left and right cuts.

We begin our attack by defining the analytic function

$$F(s;z)=\frac{1}{\pi}\int_{4\mu^2}^{\infty}dx\, e^{i\delta(x)}\sin\delta(x)F(x-i\epsilon)\times\left[\frac{1}{x-z}+\frac{2}{x-\frac{1}{2}(s+3\mu^2-z)}\right], \quad (15)$$

which has cuts running along the real axis from $-\infty$ to $s-5\mu^2$ and from $4\mu^2$ to $\infty$. Then $f(s;x)$ is simply related to the boundary value of $F(z)$:

$$f(s;x)=e^{2i\delta(x)}F(s;x-i\epsilon). \quad (16)$$

Again following Gourdin and Martin[13] we define

$$\Phi(s;z)=F(s;z)D(s;z), \quad (17)$$

where

$$D(s;z)=\exp\left[-\frac{z}{\pi}\int_{4\mu^2}^{\infty}\frac{\delta(x)dx}{x(x-z)}\right].$$

We then find that $\Phi(s;z)$ has only the left cut and must satisfy the integral equation

$$\Phi(s;z)=\lambda(s)+\frac{2}{\pi}\int_{4\mu^2}^{\infty}dx\frac{e^{i\delta(x)}\sin\delta(x)}{x-\frac{1}{2}(s+3\mu^2-z)}\times\frac{D(s+3\mu^2-2x+i\epsilon)}{D(x-i\epsilon)}\Phi(s;x-i\epsilon), \quad (18)$$

where $\lambda(s)$ is at this point an arbitrary function of $s$. We have not found an exact solution to this equation. If we assume, however, that the $\pi\pi$ amplitude is large only near the resonance energy $s_R$, then, as long as $s+3\mu^2$ is not close to $3s_R$, we can find an approximate solution. By removing $\Phi$ and $D$ in the numerator from the integrand, replacing $x$ in their arguments by $s_R$, and using the relation

$$\frac{1}{\pi}\int_{4\mu^2}^{\infty}dx\, e^{i\delta(x)}\sin\delta(x)\frac{G(x)}{D(x-i\epsilon)}=\sum \text{Res}\frac{G(z)}{D(z)}, \quad (19)$$

where $G(z)$ is any meromorphic function which converges strongly enough at $\infty$, we obtain

$$\Phi(s;z)=\lambda(s)\left\{1-2\left[2D^{-1}\left(\frac{s+3\mu^2-s_R+i\epsilon}{2}\right)-D^{-1}(s+3\mu^2-2s_R+i\epsilon)\right]D^{-1}\left(\frac{s+3\mu^2-z}{2}\right)\right\}. \quad (20)$$

When $s=3(s_R-\mu^2)$, the peak of $\Phi(s;x-i\epsilon)$ at $x=2s_R-s-3\mu^2$ would coincide with the peak in $F(s;x-i\epsilon)$ due to $D(x-i\epsilon)$, but, unfortunately, the approximations leading to (20) are no longer valid for this value of $s$.

The expression (20) suggests, however, that a reasonable form for $\Phi(s;x-i\epsilon)$ is

$$\Phi(s;x-i\epsilon)\approx\lambda(s)+\sum_\alpha R_\alpha(s)[x-z_\alpha(s)]^{-1}, \quad (21)$$

where $\text{Im}z_\alpha(s)>0, \text{Re}z_\alpha(s)<s-5\mu^2$ and the $z_\alpha(s)$ correspond to the location of poles of $\Phi$ on the unphysical sheet. If we assume this form we obtain, again using (19),

$$\begin{aligned}H(s;s_1,s_2,s_3)&=\sum_i\frac{1}{\pi}\int dx\frac{e^{i\delta(x)}\sin\delta(x)}{D(x-i\epsilon)(x-s_i-i\epsilon)}\Phi(s;x-i\epsilon),\\&=\sum_i\left\{\frac{\lambda(s)}{D(s_i+i\epsilon)}+\sum_\alpha\frac{R_\alpha(s)}{z_\alpha(s)-s_i}\left[\frac{1}{D(z_\alpha(s))}-\frac{1}{D(s_i+i\epsilon)}\right]\right\},\\&=\sum_i\frac{1}{D(s_i+i\epsilon)}\left\{\lambda(s)-\sum_\alpha\frac{R_\alpha(s)}{D(z_\alpha(s))}\times\frac{D(z_\alpha(s))-D(s_i+i\epsilon)}{z_\alpha(s)-s_i}\right\}. \quad (22)\end{aligned}$$

In the resonance approximation, when $D(x+i\epsilon)\sim s_R-x-i\Gamma$, the factor in curly brackets is independent of $s_i$ and we can write

$$H(s;s_1,s_2,s_3)\approx h(s)\sum_i D^{-1}(s_i+i\epsilon). \quad (23)$$

Even if we could solve the integral equation for $\Phi$, the solution, and, therefore, $h(s)$, would depend upon the function $\lambda(s)$ which cannot be determined from our model. If, as suggested previously by the author,[5] the connected part of the partial wave amplitude for the process $3\pi\to3\pi$ can be represented as

$$A(s;s_1,s_2,s_3;s_1',s_2',s_3')\approx\sum_{i,j}D^{-1}(s_i+i\epsilon)a(s)D^{-1}(s_j'+i\epsilon), \quad (24)$$

FIG. 2. Diagrams representing an iterative solution of the model.

then $h(s)$ will satisfy a linear integral equation with a kernel determined by $a(s)$. Since no satisfactory method for calculating $a(s)$ has yet been found, we will not discuss this problem further here except to note that although $a(s)$ is not itself directly related to any experimental results it may be useful in relating all processes in which an $I=0, J=1-, G=-1$, state is produced. We expect, for example, that $a(s)$ will have a bump at $s\approx m_\omega^2$, the square of the mass of the $\omega$ resonance, with corresponding bumps in $h(s)$ and the reduced amplitudes for other processes in which three pions are produced.

## IV. DISCUSSION

The spirit of our model is similar to that of Peierls and Tarski.[8] We consider the simplest structure which contains the features which we regard as essential: overlapping cuts which allow resonances in different channels to influence one another and a form which, when iterated, will correspond to a sum of diagrams of the kind shown in Fig. 2. It should be emphasized that we have not included contributions from diagrams, such as that shown in Fig. 3, which lead to anomalous singularities. The neglect of these diagrams is probably the weakest point of our model, for these diagrams could contribute terms with strong dependence upon all three independent variables while such terms could not possibly be described by Eqs. (7) or (23).

Accepting these limitations, our model does show that, even when there are overlapping cuts, if the two-particle interaction is dominated by a single sharp resonance then the amplitude may be written in the form (7). Each term of the sum in (7) contains a single final-state interaction as a separable factor, just as if there were a strong final-state interaction in only one channel. The discrepancy between this result and the exact solution for the production amplitude in the Lee model[9,10] is only apparent. In the Lee model there are only two independent variables since, taking $m_N=m_V$, $\omega=\omega_1+\omega_2$, where $\omega$ is the total energy and $\omega_1$ and $\omega_2$ are the energies of the two $\theta$ particles. The production amplitude is essentially of the form

$$A(\omega;\omega_1,\omega_2)=a(\omega)D^{-1}(\omega_1+i\epsilon)D^{-1}(\omega_2+i\epsilon), \quad (25)$$

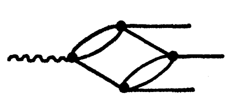

FIG. 3. A diagram not included in the model.

where $D$ is the denominator function for $N\theta$ scattering. We can, however, rearrange (25):

$$A(\omega;\omega_1,\omega_2)=a(\omega)[D(\omega_1+i\epsilon)+D(\omega_2+i\epsilon)]^{-1} \times[D^{-1}(\omega_1+i\epsilon)+D^{-1}(\omega_2+i\epsilon)]. \quad (26)$$

Then, when the two resonances come close to overlapping, $D(\omega_1+i\epsilon)+D(\omega_2+i\epsilon)\sim 2\omega_R-\omega-2i\Gamma$ and

$$A(\omega;\omega_1,\omega_2)\approx a'(\omega)[D^{-1}(\omega_1+i\epsilon)+D^{-1}(\omega_2+i\epsilon)], \quad (27)$$

which has essentially the same form as (23).

Other authors[4,19] have postulated that, in analogy with (25), $H$ should have the form

$$H(s;s_1,s_2,s_3) =h(s)D^{-1}(s_1+i\epsilon)D^{-1}(s_2+i\epsilon)D^{-1}(s_3+i\epsilon), \quad (28)$$

which would lead to spectacular enhancements where the resonance bands cross. As mentioned above, our model could not produce such a form depending strongly upon all three variables, while, in principle, this dependence could arise from anomalous diagrams such as that shown in Fig. 3. This form seems unlikely, however, when one considers that, if one pair of pions is resonating, the effects of recoil on both energy and angular momentum would seem to preclude strong resonances in the other two channels. Some idea of these recoil effects can be obtained by an examination of the integral equations in our model.

In view of the theoretical uncertainties in the treatment of three-particle states in general, and overlapping resonances in particular, an experimental study of the process $e^+ + e^- \to 3\pi$ seems particularly desirable. Some experimental results on three-pion states resulting from protonium annihilation have appeared[20] since the body of this paper was written. For this process one has to make assumptions about the quantum numbers of the initial state, but the data seem to be reasonably well described by a formula suggested by Bouchiat and Flamand,[21] equivalent to our Eq. (23), which assumes an initial state having the same quantum numbers as an isoscalar photon.

## ACKNOWLEDGMENTS

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